DETERMINING THERMOELASTIC STRESSES IN A PLATE

A. V. Sudakov and A. S. Trofimov

UDC 539.30:536.242

A method for approximate determination of thermoelastic stresses in an infinite plate is presented. With minimal complication of the problem, the proposed method yields results that deviate from the exact solutions within the limits of computational accuracy.

Determination of the thermoelastic stresses in structural elements often reduces to investigating the stresses in an infinite plane wall in the absence of deflection. If the plate is insulated on one side, the maximum stresses σ usually appear at the surface facing the heat-transport medium, and are represented by the familiar expression [1]

$$\sigma(t,\ x=h)=\frac{\alpha E}{1-\nu}\left[\frac{1}{h}\int_0^h T(t,\ x)\,dx-T(t,\ x=h)\right]=\frac{\alpha E}{1-\nu}\,[T_{av}-T_s]. \quad (1)$$

The difficulty in employing (1) lies in finding the unsteady distribution of the temperature field T(t,x).

The analytic methods are cumbersome, since they lead to infinite series (see, for example, [2, 5]). Numerical methods require the use of a digital computer. Simplified solutions are often required for practical purposes. Quasistationary methods are the most common [3]. They are simple in computational procedure, but deviate considerably from the exact solutions. We might refer to them as first-order approximations.

Let us look at a second-order approximation, which gives a more exact solution with minimal complications of the problem. The temperature field in a plate (Fig. 1) is described by the equation

$$\frac{\partial T}{\partial \tau}=\frac{\partial^2 T}{\partial z^2}+q(\tau),\quad \tau\geqslant 0,\ 0\leqslant z\leqslant 1; \quad (2)$$

$$z=0,\ \frac{\partial T}{\partial z}=0; \quad (2a)$$

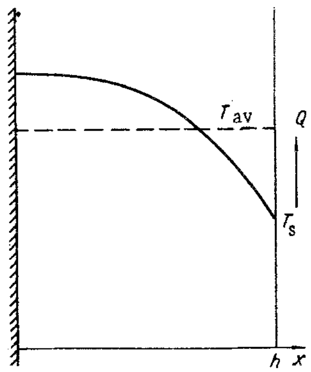

Fig. 1. Temperature field in plate.

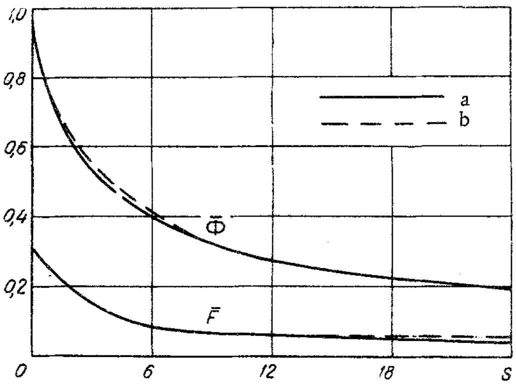

Fig. 2. Exact (a) and approximate (b) transfer functions in region of imaginary frequencies.

Energy Physics Institute, Obninsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 2, pp. 350-353, August, 1969. Original article submitted August 24, 1968.

$$z=1, \quad \frac{\partial T}{\partial z} = -\beta(\tau)(T_s - \Theta); \tag{2b}$$

$$\tau=0, \quad T=0. \tag{2c}$$

We assume that the release of heat does not depend on the spatial coordinate (although this is not a fundamentally necessary assumption).

After integrating (2) with respect to z from 0 to 1, and using the boundary conditions, we obtain

$$\frac{\partial T_{av}}{\partial \tau} + \beta(\tau) T_{\text{п}} = \beta(\tau)\Theta + q(\tau), \tag{3}$$

$$\tau=0, \quad T_{av}=0. \tag{2d}$$

The second relationship between T_{av} and T_s can be established from the same equation (2), but with the aid of a boundary condition of the first kind:

$$z=1 \quad T=T_s(\tau). \tag{2e}$$

We use a Laplace transform to solve (2), (2a), (2c), (2e) and obtain

$$\overline{T}_{av} = \overline{T}_s \overline{\Phi} + \overline{q}\overline{F}, \tag{4}$$

where $\overline{\Phi} = \text{th}\sqrt{s}/\sqrt{s}$; $\overline{F} = 1/s[1 - \text{th}\sqrt{s}/\sqrt{s}]$ are the transfer functions for the temperature and the heat release. We replace the exaft transfer functions by approximate rational functions. One of the simplest ways of doing this is to synthesize a system in the region of imaginary frequency responses [4], i.e., for real values of the parameter ($0 \le s \le \infty$). Figure 2 shows the functions $\overline{\Phi}(s)$ and $\overline{F}(s)$. The approximate values of the transfer functions are represented as

$$\overline{\Phi}_{ap} = \frac{1+Bs}{1+As+Cs^2}, \qquad \overline{F}_{ap} = E\,\frac{1+Ds}{1+As+Cs^2}. \tag{5}$$

Figure 2 shows $\overline{\Phi}_{ap}$ and $\overline{F}_{ap}$ for the following parameter values:

$$A=0.43, \; B=0.1, \; C=0.01, \; D=1/30, \; E=1/3, \tag{6}$$

selected on the basis of the recommendations given in [4].

By using the rational form of (5), we can obtain a second-order ordinary differential equation with constant coefficients for the unknown temperatures T_{av} and T_s; together with (1), (3), this yields a closed system that solves the problem:

$$\begin{gathered}
\sigma = \frac{\alpha E}{1-\nu}[T_{av} - T_s), \\
\frac{dT_{av}}{d\tau} + \beta(\tau) T_s = \beta(\tau)\Theta + q, \\
\frac{d^2T_{av}}{d\tau^2} + 43\frac{dT_{av}}{d\tau} = 100T_{av} = 10\frac{dT_s}{d\tau} + 100T_s + 1{,}11\frac{dq}{d\tau} + 33{,}3q \\
\text{for } \tau=0 \quad T_s = T_{av} = 0, \; \frac{dT_{av}}{d\tau} = q(0_+).
\end{gathered} \tag{7}$$

If we wish to allow for the way in which the heat-transfer coefficient depends on the time $\beta(\tau)$, it is best to solve the problem in the form (7). If we take β = const, the system becomes simpler. Here we must eliminate T_{av} and T_s from (7); we then obtain an equation in the unknown stresses σ:

$$\frac{d^2\sigma}{d\tau^2} + a\frac{d\sigma}{d\tau} + b\sigma = e\Theta + d\frac{d\Theta}{d\tau} - c\frac{d^2\Theta}{d\tau^2} + lq + k\frac{dq}{d\tau} - f\frac{d^2q}{d\tau^2}, \tag{8}$$

where

$$a = \frac{10+4.3\beta}{1+0.1\beta}; \quad b = \frac{10\beta}{1+0.1\beta}; \; c = \frac{\alpha E}{1-\nu}\,\frac{\beta}{10+\beta}; \; d = \frac{\alpha E}{1-\nu}\,\frac{3.33\beta}{1+0.1\beta};$$

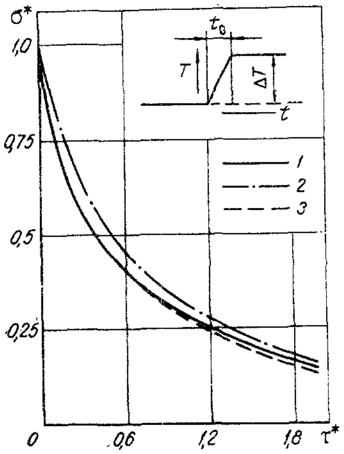

Fig. 3. Dependence of σ^*_{max} on Fourier criterion: 1) exact solution; 2) solution by Eq. (9); 3) solution by Eq. (10).

$$e = \frac{\alpha E}{1-\nu}\,\frac{20\beta}{1+0.1\beta};\quad f = \frac{\alpha E}{1-\nu}\,\frac{0.11}{10+\beta};$$

$$k = \frac{\alpha E}{1-\nu}\,\frac{0.3+1.11\beta}{10+\beta};\quad l = 3.33\,\frac{\alpha E}{1-\nu}\,\frac{6+\beta}{1+0.1\beta}.$$

Let us now compare our solution with the known exact solution [5] and quasistationary solution [3] for a linear variation (see Fig. 3) in the temperature $\Theta(\tau)$ of the heat-transport medium, and $q = 0$. When $\beta \to \infty$ (the most severe case), we obtain different values of the maximum dimensionless stresses, depending on the rate of change of the temperature, $d\Theta/d\tau = \Delta T/\tau^*$

The "parabolic" quasistationary approximation [3] gives

$$\sigma^*_{max} = \frac{1}{3\tau^*}\,[1 - \exp(-3\tau^*)]. \tag{9}$$

From the second-order approximation (8) we obtain

$$\sigma^*_{max} = \frac{0.316}{\tau^*}\,[1.0155 - 0.0155\exp(-40.5\tau^*) - \exp(-2.5\tau^*)]. \tag{10}$$

Figure 3 shows σ^*_{max} as a function of τ^*; the values were calculated from the approximate formulas given; the exact values were found from the Fritz solution [5].

NOTATION

σ	is the stress;
t	is the time;
$\tau = ta/h^2$	is the dimensionless time;
x	is the coordinate normal to the plate surface;
T(x, t)	is the plate temperature;
T_{av}	is the average plate temperature;
T_s	is the temperature of the plate surface;
Θ	is the temperature of the heat-transport medium;
h	is the plate thickness;
E	is the modulus of elasticity;
ν	is the Poisson ratio;
α	is the coefficient of linear expansion;
$\beta = kh/\lambda$, $\tau^* = t_0 a/h^2$, $q = h^2 q_V/\lambda$;	
a	is the thermal diffusivity;
q_V	is the bulk heat release;
q	is the heat-transfer coefficient.

LITERATURE CITED

1. S. P. Timoshenko, Strength of Materials [in Russian], Gostekhizdat (1946).
2. Yu. E. Bogdasarov, Atomnaya Énergiya, No. 5 (1960).
3. Yu. E. Bogdasarov and I. A. Kuznetsov, Inzh.-Fiz. Zh., 12, No. 3 (1967).
4. N. A. Orurk, New Methods for Synthesizing Linear and Certain Nonlinear Dynamic Systems [in Russian], Nauka (1965).
5. R. Fritz, Trans. ASME, 76, No. 6 (1954).